

Backreaction in trans-Planckian cosmology: renormalization, trace anomaly and selfconsistent solutions

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We analyze the semiclassical Einstein equations for quantum scalar fields satisfying modified dispersion relations. We first discuss in detail the renormalization procedure based on adiabatic subtraction and dimensional regularization. We show that, contrary to what expected from power counting arguments, in $3 + 1$ dimensions the subtraction involves up to the fourth adiabatic order even for dispersion relations containing higher powers of the momentum. Then we analyze the dependence of the trace of the renormalized energy momentum tensor with the scale of new physics, and we recover the usual trace anomaly in the appropriate limit. We also find selfconsistent de Sitter solutions for dispersion relations that contain up to the fourth power of the momentum. Using this particular example, we also discuss the possibility that the modified dispersion relation can be mimicked at lower energies by an effective initial state in a theory with the usual dispersion relation.

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I. INTRODUCTION

The expansion of the Universe can act as a cosmological microscope. Scales of interest today could have been sub-Planckian at the first stages of inflation, and it has been therefore speculated that there could be signatures of trans-Planckian physics in the evolution of the universe and/or in the inhomogeneities of the CMBR [1, 2]. It is of course very difficult to address these issues without knowing the physics at Planck scale. A plausible approach is to test the robustness of inflationary predictions under different changes that could be attributable to the unknown new physics. For example, it has been argued that one of the effects could be the modification of the dispersion relations for the modes of the quantum fields with momenta larger than a given scale M_c , as suggested by loop quantum gravity [3] or by the unavoidable interaction with gravitons [4]. It is therefore of interest to study the consequences induced by such modifications into the dynamics of the scale factor of the Universe. This is the main purpose of the present work. We will consider a quantum scalar field with a modified dispersion relation on a Robertson Walker background, and analyze the Semiclassical Einstein Equations (SEE) in order to test whether the physics at very high scales may leave an imprint at lower scales or not.

The source of the SEE is the mean value of the energy momentum tensor of the quantum fields, $\langle T_{\mu\nu} \rangle$, which is formally a divergent quantity. For quantum fields with usual dispersion relations, the covariant renormalization procedure is very well known [5, 6, 7]. For Robertson Walker metrics in n dimensions, one can compute the energy momentum tensor in the so called adiabatic approximation [8, 9, 10], and define the renormalized one as

$$\langle T_{\mu\nu} \rangle_{ren} = \langle T_{\mu\nu} \rangle - \langle T_{\mu\nu} \rangle^{(0)} \dots - \langle T_{\mu\nu} \rangle^{(2[n/2])} \quad (1)$$

where $[n/2]$ is the integer part of $n/2$. $\langle T_{\mu\nu} \rangle^{(j)}$ denotes the terms of adiabatic order j of $\langle T_{\mu\nu} \rangle$ (i.e., the terms containing j derivatives of the metric) and once regularized is proportional to a geometric tensor.

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This procedure is equivalent to a redefinition of the gravitational bare constants of the theory. In order to give sense to the divergent quantities appearing in the equation above it is necessary to use a regularization method, a very useful one being dimensional regularization. In $3 + 1$ dimensions divergences arise up to the fourth adiabatic order, and thus it is necessary to include in the gravitational part of the theory terms quadratic in the curvature. In toy $1 + 1$ -dimensional models, the fourth adiabatic order is finite, and the renormalization only involves the subtraction of the zeroth and second adiabatic orders.

The SEE have also been considered for quantum fields with modified dispersion relations [11, 12]. However, these previous works do not include a proper treatment of the divergences, and the infinities are removed by simply neglecting the zero point energy of each Fourier mode of the quantum fields. This approach is not equivalent to a redefinition of the bare constants of the theory, and therefore is not fully justified.

The extension of the adiabatic renormalization for generalized dispersion relations was first considered by us in Ref.[13], where we computed the mean value of the energy momentum tensor up to the second adiabatic order. Power counting arguments suggest that, for dispersion relations containing k^4 or higher powers of the momentum, it would be enough to subtract up to the second adiabatic order in $3 + 1$ dimensions and up to the zeroth adiabatic order in $1 + 1$ dimensions. However, a more careful analysis in $1 + 1$ dimensions showed that this is not the case [14]: indeed, the quantities to be subtracted must be expressed in terms of geometric tensors in n dimensions, and the renormalization should be performed before the limit $n \rightarrow 2$ is taken. In $1 + 1$ dimensions $G_{\mu\nu}$ is proportional to $n - 2$, since it results from the variation of the would be Gauss-Bonnet invariant in $n = 2$. Thus, the mean value of the energy momentum tensor is written as $\langle T_{\mu\nu} \rangle^{(2)} = c_n G_{\mu\nu}$ where the constant c_n contains a pole at $n = 2$. Therefore, the second adiabatic order should also be subtracted in $1 + 1$, although when explicitly computed it is a finite quantity. In this way one obtains the correct trace anomaly and the results are continuous in the limit $M_c \rightarrow \infty$ [14]. In this paper we will show that the same situation arises in $3 + 1$ dimensions with the fourth adiabatic order. The tensor that results from the variation of the would be Gauss-Bonnet invariant at $n = 4$ is proportional to $n - 4$, and therefore $\langle T_{\mu\nu} \rangle^{(4)}$, although finite, contains a pole when expressed in terms of geometric tensors in n dimensions. The conclusion will be that the subtraction must also include the fourth adiabatic order, whatever the dispersion relation.

As the calculations of the fourth adiabatic order are technically rather involved, for the benefit of the reader we would like to describe here with some detail the organization of the paper, our main conclusions, and the relation with previous works on the subject. In Section 2 we present the Lagrangian and energy momentum tensor of the scalar field with modified dispersion relations, as well as expressions for the geometric tensors in n -dimensional Robertson Walker spacetimes that will be used along the rest of the paper. Section 3 describes the WKB mode functions for the scalar field, up to the fourth adiabatic order, and for generic dispersion relations. For the usual dispersion relation, the fourth adiabatic WKB mode functions have been computed previously in Ref.[10], while for generic dispersion relations, they have been computed up to the second adiabatic order in Ref.[13]. In Section 4 we use the WKB mode functions to construct the regularized $\langle T_{\mu\nu} \rangle^{(j)}$, with $j = 0, 2, 4$ in n dimensions. These results generalize our previous ones for $j = 0, 2$, and allow us to discuss one of the main points of this paper, which is the necessity to subtract up to the fourth adiabatic order in $n = 4$. The calculations are relatively straightforward but require a lot of algebra, so we relegate the details to the Appendix A. In Section 5, and as a warm up, we analyze the renormalization of the stress tensor in $n = 2$. We compute explicitly the trace of the energy-momentum tensor in de Sitter spacetime, and show that it reproduces the usual value as $M_c \rightarrow \infty$. Section 6 deals with the renormalization of $\langle T_{\mu\nu} \rangle$ in $n = 4$. Once more, as an example, the trace of the energy-momentum tensor is computed in de Sitter space for a massless conformally coupled quantum field, and the usual trace anomaly is recovered in the limit $M_c \rightarrow \infty$.

In Section 7 we present a concrete application of the formalism developed in previous sections. We first show that in de Sitter space, for any dispersion relation, there exists a one complex family of quantum states for which the unrenormalized energy momentum tensor is proportional to $g_{\mu\nu}$. One member of the family corresponds to a renormalizable state, and for this particular state it is possible to obtain selfconsistent de Sitter solutions, as explicitly shown for massless conformally coupled fields. We find that, for values of the cosmological constant smaller than a critical value, the SEE admit two different de Sitter solutions, a

perturbative one, close to the classical solution, and a nonperturbative one, with a very high curvature.

In several previous works [15, 18], it has been argued that the trans-Planckian physics could be taken into account by considering an effective field theory with usual dispersion relations, in which the effects of the new physics is encoded in the state of the field modes when they leave the sub-Planckian regime. In Section 8 we discuss the possibility of simulate the modified dispersion relations by an effective initial quantum state at the level of the SEE, i.e. to discuss the backreaction effects.

Throughout the paper we set $c = 1$ and adopt the sign convention denoted $(+++)$ by Misner, Thorne, and Wheeler [19].

II. THE MODEL

We consider a free quantum scalar field ϕ with modified dispersion relation propagating in a curved space-time with a classical spatially flat FRW metric given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \equiv -(u_\mu dx^\mu)^2 + \perp_{\mu\nu} dx^\mu dx^\nu = C(\eta)[-d\eta^2 + \delta_{ij} dx^i dx^j] \quad (2)$$

where $\mu, \nu = 0, 1 \dots n-1$ (with n the space-time dimension), $C^{1/2}(\eta)$ is the scale factor given as a function of the conformal time η , the vector field $u_\mu \equiv C^{1/2}(\eta)\delta_\mu^\eta$, and $\perp_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu$ coincides with the spatial metric as defined by an observer comoving with u_μ .

The classical action for the scalar field can be written as [11]:

$$S_\phi = \int d^n x \sqrt{-g} (\mathcal{L}_\phi + \mathcal{L}_{cor} + \mathcal{L}_u), \quad (3)$$

where $g = \det(g_{\mu\nu})$, \mathcal{L}_ϕ is the standard Lagrangian of a free scalar field

$$\mathcal{L}_\phi = -\frac{1}{2} [g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + (m^2 + \xi R) \phi^2], \quad (4)$$

with R the Ricci scalar, \mathcal{L}_{cor} is the corrective lagrangian that gives rise to a generalized dispersion relation

$$\mathcal{L}_{cor} = - \sum_{s,p \leq n} b_{sp} (\mathcal{D}^{2s} \phi) (\mathcal{D}^{2p} \phi), \quad (5)$$

with $\mathcal{D}^2 \phi \equiv \perp_\mu^\lambda \nabla_\lambda \perp_\nu^\mu \nabla^\nu \phi$ (where ∇_μ is the covariant derivative corresponding to the metric $g_{\mu\nu}$ and $\perp_\mu^\lambda \equiv g^{\lambda\nu} \perp_{\mu\nu}$), and \mathcal{L}_u describes the dynamics of the additional degree of freedom u^μ whose explicit expression is not necessary for our present purposes.

The generalized dispersion relation takes the form

$$\omega_k^2 = k^2 + C(\eta) \left[m^2 + 2 \sum_{s,p} (-1)^{s+p} b_{sp} \left(\frac{k}{C^{1/2}(\eta)} \right)^{2(s+p)} \right], \quad (6)$$

where b_{sp} are arbitrary coefficients, with $p \leq s$.

The Fourier modes χ_k corresponding to the scaled field $\chi = C^{(n-2)/4} \phi$ satisfy

$$\chi_k'' + [(\xi - \xi_n)RC + \omega_k^2] \chi_k = 0, \quad (7)$$

with the usual normalization condition

$$\chi_k \chi_k'^* - \chi_k' \chi_k^* = i. \quad (8)$$

Here primes stand for derivatives with respect to the conformal time η , and in the conformal coupling case we have $\xi = \xi_n \equiv (n-2)/(4n-4)$, while $\xi = 0$ corresponds to minimal coupling.

On the other hand, the gravitational action is given by

$$S_G = \frac{1}{16\pi G_N} \int d^n x \sqrt{-g} (R - 2\Lambda) - \frac{1}{2} \int d^n x \sqrt{-g} (\alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}), \quad (9)$$

where $R_{\mu\nu\rho\sigma}$ is the curvature tensor, $R_{\mu\nu} = R_{\mu\rho\nu}^\rho$, and Λ , G_N , α , β , and γ are bare parameters which are to be appropriately chosen to cancel the corresponding divergences in the $\langle T_{\mu\nu} \rangle$ derived from S_ϕ . It is well known that in $n = 4$ dimensions, the quadratic terms in Eq.(9) are necessary for renormalizing the effective theory in the case of scalar fields with the usual dispersion relation. Moreover, as we will show, they are also necessary for any of the generalized dispersion relations of the type given by Eq.(6).

In the frame defined by the vector field $u_\mu = C^{1/2}(\eta)\delta_\mu^\eta$, we can write the SEE as

$$\frac{1}{8\pi G_N} (G_{\mu\nu} + \Lambda g_{\mu\nu}) + \alpha H_{\mu\nu}^{(1)} + \beta H_{\mu\nu}^{(2)} + \gamma H_{\mu\nu} = \langle T_{\mu\nu} \rangle, \quad (10)$$

where $\langle T_{\mu\nu} \rangle$ is the expectation value of the energy momentum tensor of the scalar field which in this frame satisfies $\langle T^{\mu\nu} \rangle_{;\mu} = 0$, $G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu} R/2$ is the Einstein tensor, and

$$H_{\mu\nu}^{(1)} = 2R_{;\mu\nu} - 2g_{\mu\nu}\Box R + \frac{1}{2}g_{\mu\nu}R^2 - 2RR_{\mu\nu}, \quad (11a)$$

$$H_{\mu\nu}^{(2)} = R_{;\mu\nu} - \frac{1}{2}g_{\mu\nu}\Box R - \Box R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma} - 2R^{\rho\sigma}R_{\rho\mu\sigma\nu}, \quad (11b)$$

$$H_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R_{\rho\delta\sigma\gamma}R^{\rho\delta\sigma\gamma} - 2R_{\mu\rho\delta\sigma}R_\nu{}^{\rho\delta\sigma} - 4\Box R_{\mu\nu} + 2R_{;\mu\nu} + 4R_{\mu\sigma}R^\sigma{}_\nu - 4R^{\rho\sigma}R_{\rho\mu\sigma\nu}. \quad (11c)$$

The particular combination

$$H_{\mu\nu} + H_{\mu\nu}^{(1)} - 4H_{\mu\nu}^{(2)} \equiv H_{\mu\nu}^{(3)}(n-4), \quad (12)$$

which comes from the variation of the would be Gauss-Bonnet topological invariant at $n = 4$, will be very important for our discussion about the renormalizability of the theory in four dimensions.

For the conformally flat metric that we are considering, the covariantly conserved tensors in Eq.(11) are not independent,

$$2H_{\mu\nu}^{(1)} + (n-1)[(n-2)H_{\mu\nu} - 4H_{\mu\nu}^{(2)}] = 0. \quad (13)$$

Therefore, it is enough to work with $H_{\mu\nu}^{(1)}$ and $H_{\mu\nu}^{(3)}$, whose non trivial components are given by

$$H_{\eta\eta}^{(1)} = -\frac{(n-1)^2}{C} \left[\mathcal{H}\mathcal{H}'' + \frac{(n-4)}{2}\mathcal{H}^2\mathcal{H}' - \frac{\mathcal{H}'^2}{2} + \frac{(n-10)(n-2)}{32}\mathcal{H}^4 \right], \quad (14a)$$

$$H_{11}^{(1)} = \frac{2(n-1)}{C} \left[\mathcal{H}''' + \mathcal{H}'^2 \left(\frac{1}{4} + \frac{3}{4}(n-4) \right) + \mathcal{H}\mathcal{H}'' \left(-\frac{1}{2} + (n-4) \right) \right. \\ \left. + \mathcal{H}'\mathcal{H}^2 \left(\frac{3}{8}(n-4)(n-6) - \frac{3}{2} \right) + \mathcal{H}^4 \left(\frac{3}{16} + \frac{(n-4)}{64}(n^2 - 13n + 28) \right) \right], \quad (14b)$$

$$H_{\eta\eta}^{(3)} = -\frac{(n-1)(n-2)(n-3)}{32C} \mathcal{H}^4, \quad (14c)$$

$$H_{11}^{(3)} = \frac{(n-2)(n-3)}{4C} \left[\mathcal{H}'\mathcal{H}^2 + \frac{(n-5)}{8}\mathcal{H}^4 \right], \quad (14d)$$

with $H_{11}^{(1,3)} = H_{22}^{(1,3)} = \dots = H_{(n-1)(n-1)}^{(1,3)}$, and $\mathcal{H} \equiv C'/C$.

The nontrivial components of the Einstein tensor are

$$G_{\eta\eta} = \frac{(n-1)}{4} \frac{(n-2)}{2} \mathcal{H}^2, \quad (15a)$$

$$G_{11} = G_{22} = \dots = G_{(n-1)(n-1)} = -\frac{(n-2)}{2} \left[\mathcal{H}' + \frac{(n-3)}{4} \mathcal{H}^2 \right], \quad (15b)$$

and the Ricci scalar takes the form

$$R = \frac{(n-1)}{C} \left[\mathcal{H}' + \frac{(n-2)}{4} \mathcal{H}^2 \right]. \quad (16)$$

The expectation value of the energy momentum tensor of the scalar field can be written as [20]:

$$\begin{aligned} \langle T_{\eta\eta} \rangle &= \sqrt{C} \int \frac{d^{n-1}k \mu^{\bar{n}-n}}{(2\pi\sqrt{C})^{(n-1)}} \left\{ \frac{C^{(n-2)/2}}{2} \left| \left(\frac{\chi_k}{C^{(n-2)/4}} \right)' \right|^2 + \frac{\omega_k^2}{2} |\chi_k|^2 + \xi G_{\eta\eta} |\chi_k|^2 \right. \\ &\quad \left. + \xi \frac{(n-1)}{2} \left[\frac{C'}{C} (\chi'_k \chi_k^* + \chi_k \chi_k'^*) - \frac{C'^2}{C^2} \frac{(n-2)}{2} |\chi_k|^2 \right] \right\}, \end{aligned} \quad (17)$$

$$\begin{aligned} \langle T_{11} \rangle &= \sqrt{C} \int \frac{d^{n-1}k \mu^{\bar{n}-n}}{(2\pi\sqrt{C})^{(n-1)}} \left\{ \left(\frac{1}{2} - 2\xi \right) C^{(n-2)/2} \left| \left(\frac{\chi_k}{C^{(n-2)/4}} \right)' \right|^2 + \xi G_{11} |\chi_k|^2 \right. \\ &\quad \left. + \left[\left(\frac{k^2}{n-1} \right) \frac{d\omega_k^2}{dk^2} - \frac{\omega_k^2}{2} \right] |\chi_k|^2 - \xi (\chi_k'' \chi_k^* + \chi_k \chi_k''^*) + \xi \frac{C'}{2C} (\chi'_k \chi_k^* + \chi_k \chi_k'^*) \right. \\ &\quad \left. + \xi \frac{(n-2)}{2} \left(\frac{C''}{C} - \frac{(8-n)}{4} \frac{C'^2}{C^2} \right) |\chi_k|^2 \right\}, \end{aligned} \quad (18)$$

with $T_{11} = T_{22} = \dots = T_{(n-1)(n-1)}$. Here \bar{n} is the physical space-time dimension, and μ is an arbitrary parameter with mass dimension introduced to ensure that χ has the correct dimensionality.

III. THE WKB EXPANSION

In order to compute the WKB expansion, we begin by expressing the field mode χ_k in the well-known form

$$\chi_k = \frac{1}{\sqrt{2W_k}} \exp \left(-i \int^\eta W_k(\tilde{\eta}) d\tilde{\eta} \right). \quad (19)$$

Then, the expectation values of the stress tensor can be written as

$$\begin{aligned} \langle T_{\eta\eta} \rangle &= \Omega_{n-1} \frac{\sqrt{C}}{2} \int \frac{dk k^{n-2} \mu^{\bar{n}-n}}{(2\pi\sqrt{C})^{n-1}} \left\{ \frac{[(W_k^2)']^2}{32W_k^5} + \frac{W_k}{2} + \frac{\omega_k^2}{2W_k} + \frac{(n-2)}{2} \left[\frac{C'^2(n-2)}{16W_k C^2} + \frac{C'(W_k^2)'}{8CW_k^3} \right] \right. \\ &\quad \left. + \xi \frac{G_{\eta\eta}}{W_k} - \xi \frac{(n-1)}{2} \left[\frac{C'^2}{C^2} \frac{(n-2)}{2W_k} + \frac{C'}{C} \frac{(W_k^2)'}{2W_k^3} \right] \right\}, \end{aligned} \quad (20)$$

$$\begin{aligned}
\langle T_{11} \rangle = & \Omega_{n-1} \frac{\sqrt{C}}{2} \int \frac{dk k^{n-2} \mu^{\bar{n}-n}}{(2\pi\sqrt{C})^{n-1}} \left\{ \frac{[(W_k^2)']^2}{32W_k^5} + \frac{W_k}{2} - \frac{\omega_k^2}{2W_k} + \frac{(n-2)}{2} \left[\frac{C'^2(n-2)}{16W_k C^2} + \frac{C'(W_k^2)'}{8CW_k^3} \right] \right. \\
& + \frac{k^2}{(n-1)W_k} \frac{d\omega_k^2}{dk^2} + \xi \frac{G_{11}}{W_k} + \xi \left[\frac{(W_k^2)''}{2W_k^3} - \frac{3}{4} \frac{[(W_k^2)']^2}{W_k^5} - \frac{(n-1)}{4} \frac{C'(W_k^2)'}{CW_k^3} \right] \\
& \left. + \frac{(n-2)}{2} \frac{\xi}{W_k} \left[\frac{C''}{C} - \frac{3}{2} \frac{C'^2}{C^2} \right] \right\}, \tag{21}
\end{aligned}$$

where we have defined the factor $\Omega_{n-1} \equiv 2\pi^{(n-1)/2}/\Gamma[(n-1)/2]$ coming from the angular integration.

Substitution of Eq.(19) into Eq.(7) yields

$$W_k^2 = \Omega_k^2 - \frac{1}{2} \left(\frac{W_k''}{W_k} - \frac{3}{2} \frac{W_k'^2}{W_k^2} \right), \tag{22}$$

where $\Omega_k^2 = \omega_k^2 + (\xi - \xi_n)CR$. This non-linear differential equation can be solved iteratively for W_k by assuming that it is a slowly varying function of η . In this adiabatic or WKB approximation, the adiabatic order of a term is given by the number of time derivatives.

In what follows we will denote by $^{(j)}W_k^2$ the terms in W_k^2 of adiabatic order j . We obtain straightforwardly $^{(2)}W_k^2$ replacing W_k by Ω_k on the right hand side of Eq.(22),

$$\begin{aligned}
^{(2)}W_k^2 = & (\xi - \xi_n)(n-1) \left(\frac{C''}{C} + \frac{(n-6)}{4} \frac{C'^2}{C^2} \right) \\
& - \frac{1}{4} \frac{C''}{C} \left(1 - \frac{k^2}{\omega_k^2} \frac{d\omega_k^2}{dk^2} \right) - \frac{1}{4} \frac{C'^2}{C^2} \frac{k^4}{\omega_k^2} \frac{d^2\omega_k^2}{d(k^2)^2} \\
& + \frac{5}{16} \frac{C'^2}{C^2} \left(1 - \frac{k^2}{\omega_k^2} \frac{d\omega_k^2}{dk^2} \right)^2 \\
& = \frac{\mathcal{H}^2}{16} [f^2 - 4\dot{f} + 4(\xi - \xi_n)(n-1)(n-2)] + \frac{\mathcal{H}'}{4} [f + 4(\xi - \xi_n)(n-1)], \tag{23}
\end{aligned}$$

where we have used the fact that $\tilde{\omega}_k^2 \equiv \omega_k^2/C$ depends on time only through the variable $x \equiv k^2/C$ to rewrite the right hand side in terms of the function

$$f \equiv \frac{d \ln \tilde{\omega}_k^2}{d \ln x} - 1, \tag{24}$$

and a dot means a derivative with respect to $\ln x$.

For $\bar{n} = 4$ dimensions we will also need the fourth adiabatic order, therefore, we perform one more iteration in Eq.(22). Discarding the higher order terms we arrive at

$$^{(4)}W_k^2 = \frac{1}{8\omega_k^2} \left\{ ^{(2)}W_k^2 [\mathcal{H}^2(2\dot{f} - 3f^2) - 2\mathcal{H}'f] - 5^{(2)}W_k^{2'} \mathcal{H}f - 2^{(2)}W_k^{2''} \right\}, \tag{25}$$

where we have used that $(\omega_k^2)'/\omega_k^2 = -\mathcal{H}f$ and $(\omega_k^2)''/\omega_k^2 = \mathcal{H}^2(f^2 + \dot{f}) - \mathcal{H}'f$.

In what follows it will be relevant to know the dependence with k of the different adiabatic orders. From Eqs. (22), (23) and (25), with the use of an inductive argument, it can be shown that the $2j$ -adiabatic order scales as ω_k^{2-2j} .

IV. REGULARIZED ADIABATIC STRESS TENSOR

In this Section we present the regularized adiabatic stress tensor, up to the fourth order, in n dimensions. Our goal will be to show that the different adiabatic orders are proportional to the conserved tensors $g_{\mu\nu}$, $G_{\mu\nu}$ and $H_{\mu\nu}^{(i)}$ that appear into the SEE. Therefore its subtraction will be equivalent to a redefinition of the bare gravitational constants of the theory.

In order to find the tensorial structure of the different adiabatic orders, we will perform several integrations by parts in the integrals appearing in the WKB expansion of the stress tensor, and use the fact that in dimensional regularization the integral of a total derivative vanishes [21]. We will sketch here the calculations, leaving the details to the Appendix A.

The zeroth and second adiabatic orders of the expectation value of the stress tensor can be evaluated from Eqs. (20), (21) and (23). They have been computed in Ref.[13], and are given by [22]:

$$\langle T_{\eta\eta} \rangle^{(0)} = \frac{C}{4} \frac{\Omega_{n-1} \mu^{\bar{n}-n}}{(2\pi)^{n-1}} \int_0^\infty dx x^{\frac{(n-3)}{2}} \tilde{\omega}_k, \quad (26a)$$

$$\langle T_{11} \rangle^{(0)} = \frac{C}{4} \frac{\Omega_{n-1} \mu^{\bar{n}-n}}{(2\pi)^{n-1}} \int_0^\infty dx x^{\frac{(n-3)}{2}} \frac{(f+1)}{n-1} \tilde{\omega}_k; \quad (26b)$$

$$\langle T_{\eta\eta} \rangle^{(2)} = \frac{\Omega_{n-1} \mu^{\bar{n}-n}}{4(2\pi)^{n-1}} \int_0^\infty dx x^{\frac{(n-3)}{2}} \left\{ \frac{\mathcal{H}^2}{32} (f-n+2)^2 + \xi \frac{\mathcal{H}^2}{4} (n-1)(f-n+2) + \xi G_{\eta\eta} \right\}, \quad (27a)$$

$$\begin{aligned} \langle T_{11} \rangle^{(2)} = & \frac{\Omega_{n-1} \mu^{\bar{n}-n}}{4(2\pi)^{n-1}} \int_0^\infty dx x^{\frac{(n-3)}{2}} \left\{ \frac{(f-n+2)^2}{32(n-1)} [\mathcal{H}^2(n-1) - 4\mathcal{H}'] + \frac{\mathcal{H}^2(f-n+2)}{32(n-1)} \right. \\ & \times [4\dot{f} - f^2 + (n-2)^2] + \frac{\xi \mathcal{H}^2}{8} [4\dot{f} - 2f^2 + nf + (n-2)(n-4)] - \xi \mathcal{H}'(f-n+2) + \xi G_{11} \left. \right\}. \end{aligned} \quad (27b)$$

Performing several integrations by parts and discarding surface terms, one obtains [13]:

$$\langle T_{\mu\nu} \rangle^{(0)} = -\frac{g_{\mu\nu}}{4} \frac{\Omega_{n-1} \mu^{\bar{n}-n}}{(2\pi)^{n-1}} I_0, \quad (28)$$

$$\langle T_{\mu\nu} \rangle^{(2)} = G_{\mu\nu} \frac{\Omega_{n-1} \mu^{\bar{n}-n}}{4(2\pi)^{n-1}} \left\{ \frac{I_2}{6(n-1)(n-2)} + \left(\xi - \frac{1}{6} \right) I_1 \right\}, \quad (29)$$

where we have denoted by I_i ($i=0,1,2$) the integrals given in Table I. Eqs. (28) and (29) show explicitly that the zeroth and second adiabatic orders can be absorbed into a redefinition of the bare cosmological and Newton constants respectively.

The fourth adiabatic order can be computed by following the same procedure. Starting from Eqs. (20) and (21) for $\langle T_{\eta\eta} \rangle$ and $\langle T_{11} \rangle$, we use the adiabatic expansions given in Eqs. (23) and (25) to arrive at the following expressions for the fourth adiabatic order of these expectation values:

$$\langle T_{\eta\eta} \rangle^{(4)} = \frac{\Omega_{n-1} \mu^{\bar{n}-n}}{4C(2\pi)^{(n-1)}} [\alpha_1 \mathcal{H}' \mathcal{H}^2 + \alpha_2 \mathcal{H}'' \mathcal{H} + \alpha_3 \mathcal{H}^4 + \alpha_4 \mathcal{H}'^2], \quad (30a)$$

$$\langle T_{11} \rangle^{(4)} = \frac{\Omega_{n-1} \mu^{\bar{n}-n}}{4C(2\pi)^{(n-1)}} [\beta_1 \mathcal{H}' \mathcal{H}^2 + \beta_2 \mathcal{H}'' \mathcal{H} + \beta_3 \mathcal{H}^4 + \beta_4 \mathcal{H}'^2 + \beta_5 \mathcal{H}''']. \quad (30b)$$

The coefficients α_i and β_i are given in terms of integrals that involve $\tilde{\omega}_k$ and its derivatives. As anticipated, one can find relations between them in order to show explicitly the geometric structure of the adiabatic

$I_0 = \int_0^\infty dx x^{\frac{(n-3)}{2}} \tilde{\omega}_k$	$I_5 = \int_0^\infty dx x^{\frac{(n+3)}{2}} \frac{d^3 \tilde{\omega}_k^2}{\tilde{\omega}_k^5 dx^3}$
$I_1 = \int_0^\infty dx x^{\frac{(n-3)}{2}} \frac{1}{\tilde{\omega}_k}$	$I_6 = \int_0^\infty dx x^{\frac{(n+5)}{2}} \frac{d^4 \tilde{\omega}_k^2}{\tilde{\omega}_k^5 dx^4}$
$I_2 = \int_0^\infty dx x^{\frac{(n+1)}{2}} \frac{d^2 \tilde{\omega}_k^2}{\tilde{\omega}_k^3 dx^2}$	$I_7 = \int_0^\infty dx x^{\frac{(n+5)}{2}} \left(\frac{d^2 \tilde{\omega}_k^2}{dx^2} \right)^2$
$I_3 = \int_0^\infty dx x^{\frac{(n-3)}{2}} \frac{1}{\tilde{\omega}_k^3}$	$I_8 = \int_0^\infty dx x^{\frac{(n+7)}{2}} \frac{d^5 \tilde{\omega}_k^2}{\tilde{\omega}_k^5 dx^5}$
$I_4 = \int_0^\infty dx x^{\frac{(n+1)}{2}} \frac{d^2 \tilde{\omega}_k^2}{\tilde{\omega}_k^5 dx^2}$	$I_9 = \int_0^\infty dx x^{\frac{(n+7)}{2}} \frac{d^3 \tilde{\omega}_k^2}{\tilde{\omega}_k^7 dx^3} \frac{d^2 \tilde{\omega}_k^2}{dx^2}$

TABLE I: Explicit expressions for I_i . To obtain these integrals we have made the change of variables $x \equiv k^2/C$ and we have defined $\tilde{\omega}_k = \omega_k/\sqrt{C}$.

stress tensor. To carry out this procedure, it is convenient to express all these coefficients in terms of the integrals I_i defined in Table I. For example, let us consider the coefficient α_2 ,

$$\begin{aligned}
\alpha_2 &= -\frac{1}{64} \int_0^{+\infty} \frac{dx x^{\frac{(n-3)}{2}}}{\tilde{\omega}_k^3} [4(n-1)(\xi - \xi_n) + f]^2 \\
&= -\frac{1}{4} (n-1)^2 (\xi - \xi_n)^2 I_3 - \frac{1}{8} (n-1)(\xi - \xi_n) \int_0^{+\infty} \frac{dx x^{\frac{(n-3)}{2}}}{\tilde{\omega}_k^3} f - \frac{1}{64} \int_0^{+\infty} \frac{dx x^{\frac{(n-3)}{2}}}{\tilde{\omega}_k^3} f^2. \quad (31)
\end{aligned}$$

Taking into account the definition of f (Eq.(24)) we have,

$$\int_0^{+\infty} \frac{dx x^{\frac{(n-3)}{2}}}{\tilde{\omega}_k^3} f = -\frac{2}{3} \int_0^\infty x^{\frac{(n-1)}{2}} \frac{d\tilde{\omega}_k^{-3}}{dx} - I_3 = \frac{(n-4)}{3} I_3, \quad (32a)$$

$$\begin{aligned}
\int_0^{+\infty} \frac{dx x^{\frac{(n-3)}{2}}}{\tilde{\omega}_k^3} f^2 &= -\frac{2}{5} \int_0^\infty x^{\frac{(n+1)}{2}} \frac{d\tilde{\omega}_k^{-5}}{dx} \frac{d\tilde{\omega}_k^2}{dx} + \frac{4}{3} \int_0^\infty x^{\frac{(n-1)}{2}} \frac{d\tilde{\omega}_k^{-3}}{dx} + I_3 \\
&= \frac{1}{15} (n-4)(n-6) I_3 + \frac{2}{5} I_4. \quad (32b)
\end{aligned}$$

and therefore

$$\alpha_2 = -\frac{(n-1)^2}{4} \left(\xi - \frac{1}{6} \right)^2 I_3 + \frac{(n-4)(n-1)}{1440} I_3 - \frac{I_4}{160}. \quad (33)$$

Using a similar procedure, outlined in Appendix A, it can be shown that the coefficients are related in such a way that the regularized expression for $\langle T_{\mu\nu} \rangle^{(4)}$ takes the form

$$\langle T_{\mu\nu} \rangle^{(4)} = B_1 H_{\mu\nu}^{(1)} + B_3 H_{\mu\nu}^{(3)}, \quad (34)$$

where $H_{\mu\nu}^{(1)}$ and $H_{\mu\nu}^{(3)}$ are the tensors given in Eq.(14), and

$$B_1 = \frac{\Omega_{n-1}\mu^{\bar{n}-n}}{4(2\pi)^{(n-1)}} \left\{ \frac{I_3}{4} \left[\left(\xi - \frac{1}{6} \right)^2 + \frac{(n-4)}{360(n-1)} \right] + \frac{I_4}{160(n-1)^2} \right\} \quad (35a)$$

$$B_3 = \frac{\Omega_{n-1}\mu^{\bar{n}-n}}{4(2\pi)^{(n-1)}} \left\{ \frac{I_3(n-6)}{1440(n-3)} + \frac{I_4}{(n-4)(n-3)} \left[\frac{(n+2)}{4(n-2)}(\xi - \xi_n)^2 + \frac{9(n+26)}{1440(n-1)} \right] \right. \\ \left. + \frac{3}{1440} \frac{(32I_5(n-5) + 60I_6 - 210I_7)}{(n-1)(n-2)(n-3)(n-4)} \right\}, \quad (35b)$$

where all the integrals are shown in Table I. From Eq.(34) we see that the fourth adiabatic order of the energy momentum tensor can be absorbed into redefinitions of the bare gravitational constants α , β , and γ of the SEE (10).

V. RENORMALIZATION OF THE STRESS TENSOR IN 1+1 DIMENSIONS

Knowing the dependence with k of the $2j$ -adiabatic orders, one can use power counting arguments to see which of the adiabatic orders contain divergences. In 1 + 1 dimensions, it is simple to check that for any of the dispersion relations we are considering, even for the usual one, the zeroth adiabatic order is divergent, while the higher orders are finite. This suggests that only a redefinition of the cosmological constant would be required to absorb the divergences and, then, one would define the renormalized stress tensor as $\langle T_{\mu\nu} \rangle_{ren} = \langle T_{\mu\nu} \rangle - \langle T_{\mu\nu} \rangle^{(0)}$. However, as will be explained below, this naive argument is incorrect. For the usual dispersion relation, it is known that to obtain the renormalized stress tensor it is necessary to subtract not only the zeroth adiabatic order, but also the second one [5]. As for the usual dispersion relation $I_2 = 0$, from Eq.(29) we obtain

$$\begin{aligned} \langle T_{\mu\nu} \rangle^{(2)} &= G_{\mu\nu} \frac{\Omega_{n-1}\mu^{\bar{n}-n}}{4(2\pi)^{n-1}} \left(\xi - \frac{1}{6} \right) I_1 \\ &= -\frac{G_{\mu\nu}}{2\pi} \left(\xi - \frac{1}{6} \right) \left[\frac{1}{n-2} + \ln \left(\frac{m}{2\mu} \right) + \mathcal{O}(n-2) \right]. \end{aligned} \quad (36)$$

The point is that the Einstein tensor (which results from the variation of the would be Gauss-Bonnet topological invariant at $n=2$) vanishes as $n \rightarrow 2$: $G_{\mu\nu} \propto n-2$ (see Eq.(15)). Therefore, from Eq.(36) we see that the second adiabatic order of the n -dimensional $\langle T_{\mu\nu} \rangle$ is finite in the limit $n \rightarrow 2$. However, when written in terms of $G_{\mu\nu}$ an explicit pole at $n = 2$ appears. As the pole must be absorbed into the bare constant before taking the limit $n \rightarrow 2$, a redefinition of the bare Newton constant is also required. In fact, the contribution of the second adiabatic order yields the well known trace anomaly in the case of a massless, conformally coupled field ($\xi = 0$) [5]:

$$\langle T^\mu_\mu \rangle_{ren} = -\langle T^\mu_\mu \rangle^{(2)} = \frac{R}{24\pi}. \quad (37)$$

On the other hand, for a generalized dispersion relation such that $\omega_k^2 \sim k^{2r}$ with $r > 1$, the integral I_1 is finite, but the pole appears explicitly in Eq.(29) multiplying I_2 (which is also finite). Note that the subtraction of $\langle T_{\mu\nu} \rangle^{(2)}$ can be consistently done provided that it is proportional to $G_{\mu\nu}$ and hence the pole can be absorbed into the bare Newton constant. Since the renormalization prescription must be equivalent to a redefinition of the bare constants in the effective Lagrangian of the theory, we conclude that the second adiabatic order should also be subtracted in this case. Therefore, the renormalized stress tensor is given by [14]

$$\langle T_{\mu\nu} \rangle_{ren} = \langle T_{\mu\nu} \rangle - \langle T_{\mu\nu} \rangle^{(0)} - \langle T_{\mu\nu} \rangle^{(2)}. \quad (38)$$

As a consistency check of this renormalization prescription, let us consider a massless field with $\xi = \xi_2 = 0$ in de Sitter space-time, $C(\eta) = \alpha^2/\eta^2$, and a particular dispersion relation of the form $\omega_k^2 = k^2 + 2b_{11}k^4/C(\eta)$, with $b_{11} > 0$. We will compute the trace anomaly by taking the limit in which the dispersion relation tends to the usual one ($b_{11} \rightarrow 0$). If we only subtract the zeroth adiabatic order, we have

$$\langle T_\mu^\mu \rangle - \langle T_\mu^\mu \rangle^{(0)} = -\frac{1}{\pi C} \int_0^\infty dk \left(1 - \frac{k^2}{\omega_k^2} \frac{d\omega_k^2}{dk^2} \right) \left(\omega_k^2 |\chi_k|^2 - \frac{\omega_k}{2} \right) \quad (39)$$

The modes of the field satisfy (see Eq.(7))

$$\frac{\partial^2 \chi_k}{\partial \eta^2} + \left(k^2 + \frac{2b_{11}k^4\eta^2}{\alpha^2} \right) \chi_k = 0. \quad (40)$$

With the substitution $s = (2b_{11})^{1/4} \alpha^{-1/2} k \eta$ and introducing the constant $\lambda = \alpha(2b_{11})^{-1/2}$, the equation becomes

$$\frac{\partial^2 \chi_k}{\partial s^2} + (\lambda + s^2) \chi_k = 0. \quad (41)$$

The particular solution of this equation that satisfies the normalization condition (8) and tends to the adiabatic mode of positive frequency for $|s| \rightarrow \infty$ ($\eta \rightarrow -\infty$) is given by [13]:

$$\chi_k(s) = \frac{e^{-\lambda\pi/8}}{\sqrt{k}} \left(\frac{\lambda}{2} \right)^{1/4} D_{-(\frac{1-i\lambda}{2})} [(1-i)s], \quad (42)$$

where D is the parabolic function [24, 25], $s = k\eta/\sqrt{\lambda}$ and $\lambda = \alpha/\sqrt{2b_{11}}$. After changing variables and some algebra we get

$$\langle T_\mu^\mu \rangle - \langle T_\mu^\mu \rangle^{(0)} = \frac{R}{2\pi} \int_0^\infty ds s^3 \left\{ f(\lambda, s) - \frac{\sqrt{\lambda}}{2\sqrt{\lambda + s^2}} \right\}, \quad (43)$$

where $f(\lambda, s) \equiv k|\chi_k(s)|^2$ and $R = 12\alpha^{-2}$. A numerical evaluation of this integral gives, in the limit $b_{11} \rightarrow 0$,

$$\langle T_\mu^\mu \rangle - \langle T_\mu^\mu \rangle^{(0)} \rightarrow -\frac{R}{24\pi}. \quad (44)$$

As for the case of the usual dispersion relation ($b_{11} = 0$), the trace of the stress tensor has an anomaly. However, the numerical value does not coincide with the usual one (it differs by a sign). Therefore, if we subtract only the zeroth adiabatic order, there is a discontinuity in the renormalized stress tensor as $b_{11} \rightarrow 0$. This discontinuity disappears if we also subtract the second adiabatic order. Indeed, from Eq.(29) we find, near $n = 2$,

$$\langle T_\mu^\mu \rangle^{(2)} = -\frac{R}{48\pi} ((2-n)I_1 + I_2) \mu^{2-n} \quad (45)$$

As I_1 is finite for non vanishing coefficient b_{11} , the first term does not contribute to the trace in $n = 2$. On the other hand, I_2 is independent of b_{11} , and an explicit evaluation gives

$$\langle T_\mu^\mu \rangle^{(2)} = -\frac{R}{12\pi}. \quad (46)$$

So, combining Eqs. (38), (44) and (46) we see that the usual trace anomaly (37) is recovered in the limit $b_{11} \rightarrow 0$,

$$\langle T_\mu^\mu \rangle_{ren} = \langle T_\mu^\mu \rangle - \langle T_\mu^\mu \rangle^{(0)} - \langle T_\mu^\mu \rangle^{(2)} \rightarrow \frac{R}{24\pi}. \quad (47)$$

For the sake of completeness, we compute the trace of the stress tensor (T), renormalized according to the prescription in Eq.(38), for all values of b_{11} (we recall that the limit $m \rightarrow 0$ has to be taken at the end of the calculations). In Fig. 1 we have plotted the trace T as a function of $\lambda = \alpha/\sqrt{2b_{11}}$. In this figure we see that as λ increases (b_{11} decreases) the trace approaches its anomalous value.

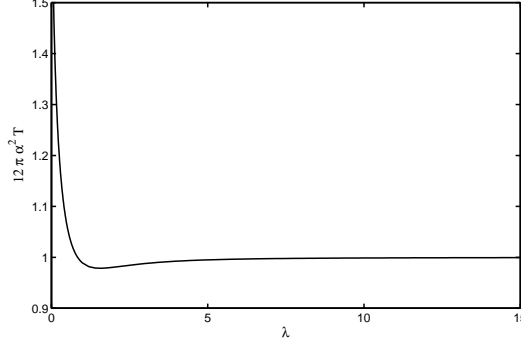


FIG. 1: The renormalized trace of the stress tensor T (with the prescription in Eq.(38)) normalized to its anomalous value as a function of $\lambda = \alpha/\sqrt{2b_{11}}$. These numerical results correspond to the massless scalar field with $\xi = 0$ and $\omega_k^2 = k^2 + 2b_{11}k^4/C(\eta)$ propagating in a two dimensional de Sitter background.

VI. RENORMALIZATION OF THE STRESS TENSOR IN 3+1 DIMENSIONS

In this Section, we will show that in 3 + 1 dimensions the fourth adiabatic order can be consistently subtracted, and that this must be done for any of the dispersion relations of the form given in Eq.(6).

In 3 + 1 dimensions, by power counting one can show that for $\omega_k^2 \sim k^{2r}$, with $r \geq 4$, all contributions of second or higher adiabatic orders are finite. The divergences come only from the zeroth order terms contained in $\langle T_{\mu\nu} \rangle$. In the cases $\omega_k^2 \sim k^6$ and $\omega_k^2 \sim k^4$, though no fourth order divergences appear, second order terms include divergent contributions, which suggests that no terms quadratic in the curvature would be necessary in the SEE. However, the situation in 3 + 1 dimensions is similar to that in 1 + 1 [23]. By using the definition of $H_{\mu\nu}^{(3)}$ given in Eq.(12), we can rewrite Eq.(34) as

$$\langle T_{\mu\nu} \rangle^{(4)} = B_1 H_{\mu\nu}^{(1)} + \frac{B_3}{(n-4)} \left[H_{\mu\nu} + H_{\mu\nu}^{(1)} - 4H_{\mu\nu}^{(2)} \right]. \quad (48)$$

As the coefficient B_3 does not vanish in four dimensions, an explicit pole at $n = 4$ appears, which must be absorbed into the bare constants of the effective gravitational action (9).

If the fourth adiabatic order is not subtracted, there would be a discontinuity in the limit in which the dispersion relation tends to the usual one. This means that the renormalized stress tensor would contain non vanishing trans-Planckian contributions, even when $M_C \rightarrow \infty$. We illustrate this point by computing the trace of the energy momentum tensor for a massless field with conformal coupling $\xi = 1/6$ and $\omega_k^2 = k^2 + 2b_{11}k^4/C(\eta)$ in de Sitter space-time, $C(\eta) = \alpha^2/\eta^2$ (as we have done for the 1 + 1 dimensional case). From Table I one can see that for the usual dispersion relation the only divergent integral appearing in the fourth order is I_3 (see Eqs. (34) and (35)). So, near $n = 4$, we have

$$\langle T_{\mu\nu} \rangle^{(4)} \rightarrow \frac{1}{2880\pi^2} \left[\frac{1}{6} {}^{(1)}H_{\mu\nu} + {}^{(3)}H_{\mu\nu} \right], \quad (49)$$

which is a well known result [5, 10]. Therefore, the usual trace anomaly is given by

$$\langle T_\mu^\mu \rangle_{ren} = -\langle T_\mu^\mu \rangle^{(4)} = -\frac{R^2}{34560\pi^2} = -\frac{1}{240\pi^2\alpha^4}. \quad (50)$$

On the other hand, for $b_{11} > 0$ all integrals are finite and can be explicitly computed. We find

$$\langle T_\mu^\mu \rangle^{(4)} = \frac{1}{120\pi^2\alpha^4}, \quad (51)$$

which is -2 times the usual trace anomaly. Note that the result is independent of b_{11} .

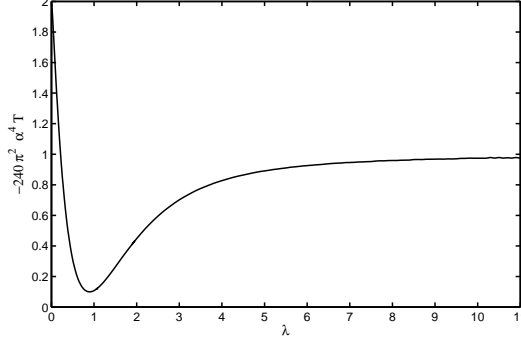


FIG. 2: The renormalized trace of the stress tensor normalized to its anomalous value as a function of $\lambda = \alpha/\sqrt{2b_{11}}$. The numerical result corresponds to the massless scalar field with $\xi = 1/6$ and $\omega_k^2 = k^2 + 2b_{11}k^4/C(\eta)$ propagating in a four dimensional de Sitter background.

The trace of the unrenormalized stress tensor with the subtraction of the zeroth and second adiabatic orders can be written in the form

$$\langle T_\mu^\mu \rangle - \langle T_\mu^\mu \rangle^{(0)} - \langle T_\mu^\mu \rangle^{(2)} = \frac{\lambda}{2\pi^2\alpha^4} \int_0^{+\infty} ds s^5 \left\{ f(\lambda, s) - \frac{\sqrt{\lambda}}{2(\lambda + s^2)^{1/2}} - \frac{\sqrt{\lambda}}{8(\lambda + s^2)^{5/2}} + \frac{5\sqrt{\lambda}}{16(\lambda + s^2)^{7/2}} \right\}. \quad (52)$$

By means of a numerical evaluation, as $b_{11} \rightarrow 0$, we obtain

$$\langle T_\mu^\mu \rangle - \langle T_\mu^\mu \rangle^{(0)} - \langle T_\mu^\mu \rangle^{(2)} \rightarrow \frac{1}{240\pi^2\alpha^4}. \quad (53)$$

Therefore, the usual trace anomaly is recovered in the limit $b_{11} \rightarrow 0$ only when the fourth adiabatic order is also subtracted:

$$\langle T_\mu^\mu \rangle_{ren} = \langle T_\mu^\mu \rangle - \langle T_\mu^\mu \rangle^{(0)} - \langle T_\mu^\mu \rangle^{(2)} - \langle T_\mu^\mu \rangle^{(4)} \rightarrow -\frac{1}{240\pi^2\alpha^4}. \quad (54)$$

The behaviour of the renormalized trace T as a function of $\lambda = \alpha/\sqrt{2b_{11}}$ is shown in Fig.2, where we see that it approaches to its anomalous value as the dispersion relation tends to the usual one.

VII. SELFCONSISTENT DE SITTER SOLUTIONS

As an application of the results presented in the previous sections, here we will study the SEE in de Sitter space-time, $C(\eta) = \alpha^2/\eta^2$, taking into account quantum effects of free scalar fields with a generalized dispersion relation (i.e., including the backreaction of the quantum fields on the spacetime dynamics).

It is well known that in de Sitter space-time the expectation value of the energy momentum tensor $\langle T_{\mu\nu} \rangle$ of a free scalar field with the usual dispersion relation is proportional to the metric $g_{\mu\nu}$ in the de Sitter invariant states [26]. As they are related by Bogoliubov transformations with momentum independent coefficients, these states form a one complex parameter family. Here, we will point out that for a generalized dispersion relation there also exists a one complex parameter family of states for which $\langle T_{\mu\nu} \rangle \propto g_{\mu\nu}$, which is just the tensorial structure of the cosmological constant term. We will then analyze the relation between the de Sitter curvature ($R = 12\alpha^{-2}$) and the cosmological constant (Λ) for a dispersion relation of the form $\omega_k^2 = k^2 + 2b_{11}k^4/C(\eta)$.

In de Sitter space-time, the field modes χ_k satisfy (see Eq.(7))

$$\frac{\partial^2 \chi_k}{\partial \eta^2} + \left(\omega_k^2(\eta) + \frac{\tilde{\mu}^2 \alpha^2}{\eta^2} \right) \chi_k = 0, \quad (55)$$

where $\tilde{\mu}^2 = m^2 + n(n-1)(\xi - \xi_n)/\alpha^2$. With the substitution $s = k\eta/\sqrt{\lambda}$, this equation can be recast as

$$\frac{\partial^2 \chi_k}{\partial s^2} + \left(\bar{\omega}^2(s) + \frac{\tilde{\mu}^2 \alpha^2}{s^2} \right) \chi_k = 0. \quad (56)$$

where $\bar{\omega}^2(s) = \omega_k^2(\eta)/k^2$ is a function of k and η only through the variable s . Let $f(s)$ and $g(s)$ be two independent solutions of this equation. Then, a field mode can be conveniently expressed as

$$\chi_k(s) = \frac{A_k}{\sqrt{k}} f(s) + \frac{B_k}{\sqrt{k}} g(s). \quad (57)$$

We now choose the coefficients $A_k = A$ and $B_k = B$ to be momentum independent. Thus, defining $\psi(s) \equiv \sqrt{k} \chi_k(s)$, the normalization condition (8) becomes

$$\psi(s) \frac{\partial \psi^*}{\partial s}(s) - \frac{\partial \psi}{\partial s}(s) \psi^*(s) = i\sqrt{\lambda}. \quad (58)$$

By introducing these particular solutions ($\chi_k(s) = \psi(s)/\sqrt{k}$) into Eqs. (17) and (18), and rescaling the integration variable one can show that $\rho = \langle T_{\eta\eta} \rangle / C$ and $p = \langle T_{11} \rangle / C$ are time independent. Therefore, for the corresponding states we have $\langle T_{\mu\nu} \rangle \propto g_{\mu\nu}$, provided that $\langle T_{\mu\nu} \rangle$ is covariantly conserved. It is remarkable that this one-parameter family of states exists for any dispersion relation.

If we choose the particular state of the family that reproduces the WKB solution as $|s| \rightarrow \infty$, the divergences in the stress tensor can be absorbed into the bare gravitational constants, and the SEE can be recast as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi N G_N \langle T_{\mu\nu} \rangle_{ren}, \quad (59)$$

where we N is the number of scalar fields and we assumed that the renormalized values of α , β and γ vanish. Using that in de Sitter space-time $R_{\mu\nu} = Rg_{\mu\nu}/4$ and $\langle T_{\mu\nu} \rangle_{ren} = Tg_{\mu\nu}/4$ (where $T \equiv \langle T^\mu_\mu \rangle_{ren}$), we have

$$-\frac{R}{4} + \Lambda = 2\pi N G_N T. \quad (60)$$

Therefore, the de Sitter metric is a consistent solution of the SEE even when the backreaction of scalar quantum fields with a generalized dispersion relation is included.

For a massless field with $\xi = 1/6$ and $\omega_k^2 = k^2 + 2b_{11}k^4/C(\eta)$ ($\tilde{\mu}^2 = 0$), Eq.(56) can be solved exactly. The solution for the modes are given in Eq.(42) (where $s = (2b_{11})^{1/4} \alpha^{-1/2} k\eta \equiv k\eta/\sqrt{\lambda}$). Note that as $b_{11} \rightarrow 0$ these field modes tend to the standard Bunch-Davies modes, defining a generalized Bunch-Davies vacuum. To get the renormalized trace T of the stress tensor we subtract up to the fourth adiabatic order. An integral expression for the unrenormalized trace with the subtraction of the zeroth and second adiabatic order is given in Eq.(52), while in Eq.(51) we have the fourth adiabatic order. Note that the function $f \equiv T/R^2$ depends on only one free parameter $b_{11}R = 6/\lambda^2$. Therefore, we have

$$\Lambda(R) = \frac{R}{4} + 2\pi N G_N T = \frac{R}{4} + 2\pi N G_N R^2 f(b_{11}R). \quad (61)$$

We evaluate $f(b_{11}R)$ numerically for different values of $b_{11}R$. The relation in Eq.(61) is shown in Fig. 3, where we have also plotted the classical relation $\Lambda = R/4$. The results are presented for $b_{11}m_{pl}^2 = 1$ and

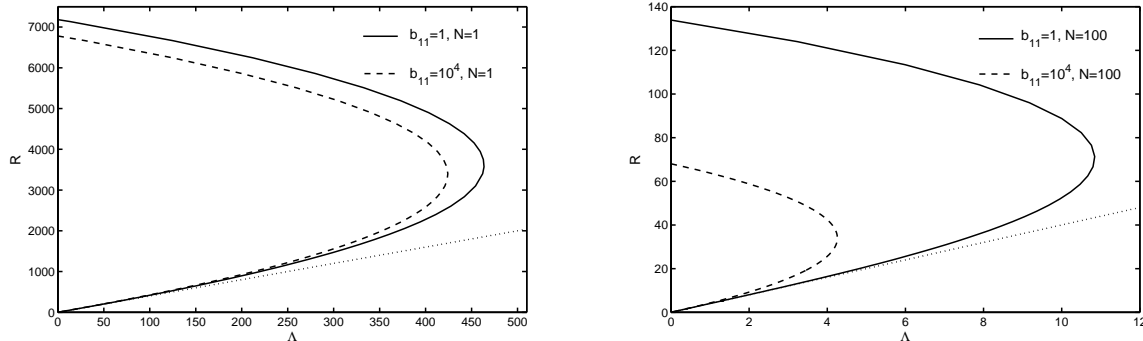


FIG. 3: The relation in Eq.(61) between the curvature of the de Sitter space-time R and the cosmological constant Λ for two values of b_{11} : In solid line $b_{11} = m_{pl}^{-2}$ and in dashed line $b_{11} = 10^4 m_{pl}^{-2}$ (where m_{pl} is the Planck mass), with $N = 1$ (on the left) and $N = 100$ (on the right). In each case, the classical relation $\Lambda = R/4$ is shown by a dotted line. These results correspond to N massless scalar fields with $\xi = 1/6$ and $\omega_k^2 = k^2 + 2b_{11}k^4/C(\eta)$ in four dimensions. Note the different scales in both axes.

10^4 (where m_{pl} is the Planck mass) for $N = 1$ and $N = 100$ (for intermediate values of the parameters the results lie in between). From these figures, we can see that for any of the values of the parameters the general features are common. There is no selfconsistent solution for large values of the cosmological constant Λ . The value of Λ above which there is no more selfconsistent solution becomes larger as b_{11} decreases. When Λ is small there are two selfconsistent solutions: one is near the classical solution $\Lambda = R/4$ while the other has a positive curvature R larger than 4Λ , even for negative values of Λ , which in general is far from being in the semiclassical regime. These results are similar to those obtained in Ref.[27] for the usual dispersion relation.

VIII. AN EFFECTIVE INITIAL STATE?

It has been suggested in the literature that the trans-Planckian effects could be taken into account in a low energy effective field theory with usual dispersion relations, by considering a generic “initial” state for the modes of the quantum field when they leave the sub-Planckian regime [15]. There is some debate about whether the trans-Planckian corrections to the power spectrum of primordial fluctuations can be consistently reproduced from a suitable choice of the initial state [16, 17, 18]. Moreover, the choice of the initial time for imposing the initial condition is a nontrivial problem. In fact, as was pointed out in Ref.[30], an inadequate choice could lead to artificial oscillations in the power spectrum. On the other hand, within this description it is difficult to quantify the backreaction effects [12].

We will discuss now whether the backreaction effects on the spacetime metric can be taken into account by considering an arbitrary initial state in the low energy effective theory. If we adopt the usual renormalization prescription in the effective theory, the divergences in $\langle S|T_{\mu\nu}|S\rangle$ can be absorbed into counterterms of the gravitational effective action if the state $|S\rangle$ coincides with the adiabatic state up to the fourth order (see for instance [28, 29]). Hence, the β_k coefficient of the Bogoliubov transformation that relate the mode function ψ_k^S (corresponding to the state $|S\rangle$) to the Bunch-Davies one ψ_k^{BD} ,

$$\psi_k^S = \alpha_k \psi_k^{BD} + \beta_k \psi_k^{BD*}, \quad (62)$$

is required vanish faster than k^{-2} as $k \rightarrow \infty$. In previous works [16, 17, 29] it has been shown that the de Sitter invariant states belong to a one parameter family, which is related to the Bunch-Davies vacuum by constant Bogoliubov coefficients. Therefore, the only renormalizable state which is de Sitter invariant is the Bunch-Davies vacuum.

In Section VII, we have pointed out that for any generalized dispersion relation, a family of states for which $\langle T_{\mu\nu} \rangle \propto g_{\mu\nu}$ also exists (as for the de Sitter invariant states in the usual theory). Thus, by choosing the member of the family that tends to the adiabatic mode of positive frequency for $|s| \rightarrow \infty$, one can obtain a generalized Bunch-Davies state for which the renormalized stress tensor is proportional to the metric tensor. Therefore, there are states for which the expectation value of the stress tensor of a scalar field with a generalized dispersion relation cannot be reproduced by any choice of the initial quantum state of a field with the standard dispersion relation, unless other renormalization scheme is employed [18]. In other words, while can obtain selfconsistent de Sitter solutions for any generalized dispersion relation, in the standard theory this is possible only for a single quantum state.

IX. FINAL REMARKS

In this paper we have presented a complete analysis of the renormalization procedure for the semiclassical Einstein equations, in theories in which the quantum scalar fields satisfy modified dispersion relations. This work generalizes the adiabatic renormalization for quantum field theory in curved spaces developed in the seventies to theories containing higher spatial derivatives of the matter fields.

We have shown that, even though power counting suggests that for this class of theories it would be enough to renormalize the cosmological and Newton's constants, in $3+1$ dimensions a consistent procedure also involves the subtraction of the fourth adiabatic order. Therefore, it is also necessary to include terms quadratic in the curvature into the gravitational action in order to absorb the divergences of the expectation value of the energy momentum tensor. This subtle point was missed in our previous work [13], and clarified in the $1+1$ dimensional case in Ref.[14].

We obtained regularized expressions for the stress tensor up to the fourth adiabatic order, and showed explicitly the geometric nature of its divergences. We also computed the trace of the renormalized tensor in de Sitter space for the 'would be conformal field' in the standard theory, and recovered the usual trace anomaly in the limit $M_c \rightarrow \infty$.

We have also shown that, whatever the dispersion relation, there exist a family of quantum states for which the mean value of the stress tensor is proportional to the metric. One member of this family (the 'generalized Bunch-Davies' vacuum), is renormalizable, and therefore it allows the existence of selfconsistent de Sitter solutions, as explicitly shown for massless, conformally coupled fields. These solutions are not present for arbitrary states in the standard theory (they only exist for the usual Bunch-Davies vacuum), and therefore this particular trans-Planckian effect can not be simulated by modifying the quantum state in an effective theory with the usual dispersion relation.

Appendix A: Regularization of $\langle T_{\mu\nu} \rangle^{(4)}$

In this Appendix we outline the procedure by which we derived Eq.(34), starting from the unrenormalized fourth adiabatic order of the stress tensor. The idea is to find relations between the different integrals that appear in the WKB expansion of the stress tensor, in order to reveal its geometric nature. The relations can be found by performing integrations by parts and by using that in dimensional regularization the integral of a total derivative vanishes [21].

From Eqs. (20) and (21), with the use of the WKB expansion, the fourth adiabatic order of $\langle T_{\eta\eta} \rangle$ and $\langle T_{11} \rangle$ can be recast as

$$\begin{aligned} \langle T_{\eta\eta} \rangle^{(4)} &= \frac{\Omega_{n-1} \mu^{\bar{n}-n}}{4C(2\pi)^{(n-1)}} \int_0^{+\infty} \frac{dx x^{(n-3)/2}}{8\tilde{\omega}_k^3} \left\{ [({}^{(2)}W_k^2)^2 - ({}^{(2)}W_k^2)' \mathcal{H} \left[\frac{f}{2} + 2(n-1)(\xi - \xi_n) \right] \right. \\ &\quad \left. - ({}^{(2)}W_k^2 \mathcal{H}^2 \left[\frac{5}{8} f^2 + (n-1)(\xi - \xi_n) \left(3f - \frac{(n-2)}{2} \right) \right]) \right\}, \end{aligned} \quad (63)$$

$$\begin{aligned}
\langle T_{11} \rangle^{(4)} = & \frac{\Omega_{n-1} \mu^{\bar{n}-n}}{4C(2\pi)^{(n-1)}} \int_0^{+\infty} \frac{dx x^{(n-3)/2}}{8\tilde{\omega}_k^3(n-1)} \left\{ {}^{(2)}W_k^{2''} [f + 4(n-1)(\xi - \xi_n)] - [{}^{(2)}W_k^2]^2 (2n-5-3f) \right. \\
& + {}^{(2)}W_k^{2'} \mathcal{H} \left[2(n-1)(\xi - \xi_n)(6f - (n-1)) - \frac{f}{2} + \frac{5}{2}f^2 \right] + {}^{(2)}W_k^2 \mathcal{H}' \left[6(\xi - \xi_n)(n-1)f + \frac{(n-2)}{2}f + f^2 \right] \\
& + {}^{(2)}W_k^2 \mathcal{H}^2 \left[(\xi - \xi_n)(n-1) \left(\frac{1}{2}(n-1)(n-2) + 9f^2 - 6\dot{f} - 3(n-1)f \right) + \frac{3}{2}f^3 + \frac{(n-7)}{8}f^2 \right. \\
& \left. \left. - f\dot{f} - \frac{(n-2)}{2}\dot{f} \right] \right\}, \tag{64}
\end{aligned}$$

where the function $f = f(x)$ is defined in Eq.(24). Here we have used Eq.(25) to write ${}^{(4)}W_k^2$ in terms of ${}^{(2)}W_k^2$ and its derivatives. By using in addition that

$${}^{(2)}W_k^{2'} = \frac{\mathcal{H}''}{4} [4(\xi - \xi_n)(n-1) + f] + \frac{\mathcal{H}\mathcal{H}'}{8} [f^2 - 6\dot{f} + 4(n-2)(n-1)(\xi - \xi_n)] + \frac{\mathcal{H}^3}{8} [2\ddot{f} - f\dot{f}], \tag{65a}$$

$$\begin{aligned}
{}^{(2)}W_k^{2''} = & \frac{\mathcal{H}'^2}{8} [f^2 - 6\dot{f} + 4(n-2)(n-1)(\xi - \xi_n)] + \frac{\mathcal{H}\mathcal{H}''}{8} [f^2 - 8\dot{f} + 4(n-2)(n-1)(\xi - \xi_n)] \\
& + \frac{\mathcal{H}^4}{8} [\dot{f}^2 + f\ddot{f} - 2\dot{f}\ddot{f}] + \frac{\mathcal{H}^2\mathcal{H}'}{8} [12\ddot{f} - 5f\dot{f}] + \frac{\mathcal{H}'''}{4} [4(\xi - \xi_n)(n-1) + f], \tag{65b}
\end{aligned}$$

we write $\langle T_{\eta\eta} \rangle^{(4)}$ and $\langle T_{11} \rangle^{(4)}$ in the form given in Eq.(30), where the coefficients α_i and β_i can be expressed as a linear combination of integrals of the form

$$J_{mnts} \equiv \int_0^\infty dx \frac{x^{\frac{(n-3)}{2}}}{\tilde{\omega}_k^3} f^m \dot{f}^n \ddot{f}^l \ddot{f}^s. \tag{66}$$

with m, n, l, s integer numbers. For example, the coefficient α_2 considered in the text (Eq.(31)) is given by

$$\alpha_2 = -\frac{1}{4}(n-1)^2(\xi - \xi_n)^2 I_3 - \frac{1}{8}(n-1)(\xi - \xi_n) J_{1000} - \frac{1}{64} J_{2000}, \tag{67}$$

where $I_3 = J_{0000}$ is defined in Table I, and from Eq.(32) we identify

$$J_{1000} = \frac{(n-4)}{3} I_3, \tag{68a}$$

$$J_{2000} = \frac{1}{15}(n-4)(n-6) I_3 + \frac{2}{5} I_4. \tag{68b}$$

It is straightforward to show that

$$\alpha_4 = -\frac{1}{2} \alpha_2. \tag{69}$$

To find a relation between α_1 and α_2 requires a little more work. The coefficient α_1 is given by

$$\alpha_1 = -\frac{3}{8}(n-1)^2(\xi - \xi_n)^2 J_{1000} - \frac{1}{16}(n-1)(\xi - \xi_n) [3J_{2000} - 2J_{0100}] - \frac{1}{128} [3J_{3000} - 4J_{1100}]. \tag{70}$$

In order to express it in terms of the integrals I_i in Table I it is useful to observe that

$$\begin{aligned}
J_{2000} &= \int_0^\infty \frac{x^{\frac{(n-3)}{2}}}{\tilde{\omega}_k^3} f \left(\frac{x}{\tilde{\omega}_k^2} \frac{d\tilde{\omega}_k^2}{dx} - 1 \right) \\
&= -J_{1000} + 2 \int_0^\infty \frac{x^{\frac{(n-1)}{2}}}{\tilde{\omega}_k^4} \frac{d\tilde{\omega}_k}{dx} f \\
&= \frac{(n-4)}{3} J_{1000} + \frac{2}{3} J_{0100}, \tag{71}
\end{aligned}$$

where the last equality follows after performing an integration by parts and discarding the surface term. Thus we have

$$3J_{2000} - 2J_{0100} = (n-4)J_{1000}, \quad (72)$$

which is one of the combination of integrals appearing in Eq.(70). In a similar way, the integral J_{3000} can be recast as

$$\begin{aligned} J_{3000} &= \int_0^\infty \frac{x^{\frac{(n-3)}{2}}}{\tilde{\omega}_k^3} \left(\frac{x}{\tilde{\omega}_k^2} \frac{d\tilde{\omega}_k^2}{dx} - 1 \right) f^2 \\ &= -J_{2000} + 2 \int_0^\infty \frac{x^{\frac{(n-1)}{2}}}{\tilde{\omega}_k^4} \frac{d\tilde{\omega}_k}{dx} f^2 \\ &= -J_{2000} + \frac{(n-1)}{3} J_{2000} + \frac{4}{3} J_{1100}, \end{aligned} \quad (73)$$

from which we have

$$3J_{3000} - 4J_{1100} = (n-4)J_{2000}. \quad (74)$$

Inserting Eqs. (68a), (72) and (74) into Eq.(70), we find

$$\alpha_1 = -\frac{1}{8}(n-1)^2(\xi - \xi_n)^2(n-4)I_3 - \frac{1}{16}(n-1)(\xi - \xi_n)(n-4)J_{1000} - \frac{1}{128}(n-4)J_{2000}, \quad (75)$$

and comparing this coefficient with α_2 (given in Eq.(67)) we see that

$$\alpha_1 = \frac{(n-4)}{2} \alpha_2. \quad (76)$$

Therefore, with the use of Eqs. (14a), (14c), (69) and (76), the component $\langle T_{\eta\eta} \rangle^{(4)}$ can be written as

$$\begin{aligned} \langle T_{\eta\eta} \rangle^{(4)} &= \frac{\Omega_{n-1} \mu^{\bar{n}-n}}{4(2\pi)^{n-1}} \left\{ -\frac{\alpha_2 H_{\eta\eta}^{(1)}}{(n-1)^2} - \frac{32H_{\eta\eta}^{(3)}}{(n-1)(n-2)(n-3)} \left[\alpha_3 - \frac{(n-10)(n-2)}{32} \alpha_2 \right] \right\} \\ &\equiv B_1 H_{\eta\eta}^{(1)} + B_3 H_{\eta\eta}^{(3)}. \end{aligned} \quad (77)$$

Since from Eqs. (31) and (68) the coefficient α_2 can be expressed in terms of the integrals I_i , Eq.(35a) for B_1 follows straightforwardly.

To show that the same equation (77) is satisfied by the component $\langle T_{11} \rangle^{(4)}$, with $H_{\eta\eta}^{(1,3)}$ replaced by $H_{11}^{(1,3)}$, we start by noting that

$$\beta_5 = -\frac{2}{(n-1)} \alpha_2, \quad (78)$$

$$\begin{aligned} \beta_2 &= \frac{(\xi - \xi_n)^2}{2} (n-1) \left[-\frac{1}{2} + (n-4) \right] I_3 + \frac{(\xi - \xi_n)}{8} [6J_{2000} - 4J_{0100} - J_{1000}] \\ &\quad + \frac{1}{64(n-1)} [6J_{3000} - 8J_{1100} - J_{2000}], \end{aligned} \quad (79)$$

$$\begin{aligned} \beta_4 &= \frac{(\xi - \xi_n)^2}{8} (n-1) [9J_{1000} + I_3] + \frac{(\xi - \xi_n)}{16} [9J_{2000} - 6J_{0100} + J_{1000}] \\ &\quad + \frac{1}{128(n-1)} [9J_{3000} - 12J_{1100} + J_{2000}]. \end{aligned} \quad (80)$$

With the use of Eqs. (68a), (72) and (74) we have

$$\beta_2 = \left[-\frac{1}{2} + (n-4) \right] \beta_5, \quad (81)$$

$$\beta_4 = \left[\frac{1}{4} + \frac{3(n-4)}{4} \right] \beta_5. \quad (82)$$

Thus $\langle T_{11} \rangle^{(4)}$ can be expressed in the form

$$\begin{aligned} \langle T_{11} \rangle^{(4)} &= \frac{\Omega_{n-1} \mu^{\bar{n}-n}}{4(2\pi)^{n-1}} \left\{ -\frac{\alpha_2 H_{11}^{(1)}}{(n-1)^2} + \left[\beta_1 - \left(\frac{3}{8}(n-4)(n-6) - \frac{3}{2} \right) \beta_5 \right] \frac{\mathcal{H}' \mathcal{H}^2}{C} \right. \\ &\quad \left. + \left[\beta_3 - \left(\frac{3}{16} + \frac{(n-4)}{64} (n^2 - 13n + 28) \right) \beta_5 \right] \frac{\mathcal{H}^4}{C} \right\} \\ &\equiv \left\{ B_1 H_{11}^{(1)} + \tilde{\beta}_1 \frac{\mathcal{H}' \mathcal{H}^2}{C} + \tilde{\beta}_3 \frac{\mathcal{H}^4}{C} \right\}. \end{aligned} \quad (83)$$

By the same procedure of integrating by parts and discarding surface terms, and after a lot of algebra, we find

$$\tilde{\beta}_3 = \frac{(n-5)}{8} \tilde{\beta}_1 = \frac{(n-5)}{2(n-2)(n-3)} B_3, \quad (84)$$

where the coefficient B_3 is given in Eq.(35a). Finally, we arrive at Eq.(34) by inserting these last relations into Eqs. (77) and (83).

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